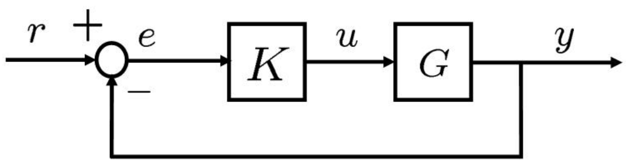
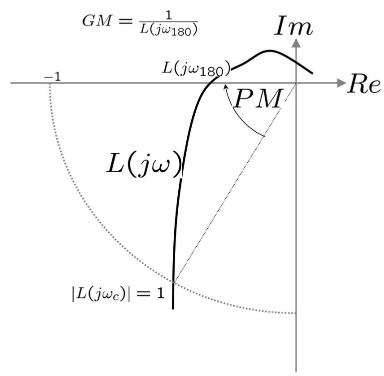
Q1 [20pts]

This question studies some of the fundamental limitations imposed on the control design due to presence of right half zeros in the plant transfer function. Consider the feedback configuration shown below:



(A)



(B)

(a) We use the notation $\|M\|_\infty$ to denote the peak value of a stable transfer function $M(s)$ where

$$\|M\|_\infty = \max_\omega |M(j\omega)| = \max_{Real(s)\geq 0} |M(s)|. \quad (1)$$

Assume that the result $\max_\omega |M(j\omega)| = \max_{Real(s)\geq 0} |M(s)|$ hold true for stable transfer function $M(s)$.

i. Find the transfer function $S(s)$ from r to e (called the sensitivity transfer fuction) from Figure (A). Show that $|S(j\omega)|$ is the reciprocal of the distance of the point $L(j\omega) = G(j\omega)K(j\omega)$ from the point $-1 + j0$ on the Nyquist plot in Figure (B). Deduce that the control design that achieves lower $\|S\|_\infty$ value achieves better relative stability (more robust) for the closed loop system. [3pts]

solution: $E(s) = R(s) - Y(s) = R(s) - G(s)K(s)E(s) \Rightarrow S(s) := E(s)/R(s) = 1/(1 + G(s)K(s))$.
The distance from $L(j\omega)$ from -1 is $|G(j\omega)K(j\omega) - (-1)| = |G(j\omega)K(j\omega) + 1| = 1/|S(j\omega)|$. Hence $|S(j\omega)|$ is the reciprocal of the distance of the point $L(j\omega) = G(j\omega)K(j\omega)$ from the point $-1 + j0$.
Smaller $\|S\|_\infty$ implies larger the distance of the Nyquist curve from -1, which implies larger relative stability.

ii. Assume that the plant $G(s)$ has a right half plane zero at z.

A. Show that $S(z) = 1$. Use equation (1) to show that it is *is not* possible to design a stabilizing controller $K(s)$ such that $|S(j\omega)| < 1$ for all ω (use that fact that $\max_\omega |M(j\omega)| = \max_{Real(s)\geq 0} |M(s)|$ if M is stable). [2pts]

solution: Since $G(z) = 0$, $S(z) = 1/(1 + G(z)K(z)) = 1$ irrespective of the control design. Since from equation (1), $\|S\|_\infty \geq |S(s)|$ for all s, therefore $\|S\|_\infty \geq |S(z)| = 1$ for any control design $K(s)$. Therefore from equation (1), $|S(j\omega)| \geq 1$ for all ω.

B. Show that it is *not* possible to design a stabilizing controller $K(s)$ such that $\|w_p(s)S(s)\|_\infty < |w_p(z)|$ where $w_p(s)$ is any stable transfer function. [2pts]

solution: Similarly, since from equation (1), $\|w_pS\|_\infty \geq |w_p(s)S(s)|$ for all s, therefore $\|w_pS\|_\infty \geq |w_p(z)S(z)| = |w_p(z)|$ for any control design $K(s)$ as $|S(z)| = 1$.

(b) Let the plant transfer function be given by

$$G(s) = \frac{3(1-2s)}{(5s+1)(10s+1)}.$$

i. Find the poles and zeros of $G(s)$. Is the plant stable? What would be your objection to using an open loop control strategy $U(s) = G^{-1}(s)R(s)$ for perfect tracking so that $Y(s) = G(s)u(s) = G(s)G^{-1}(s)R(s) = R(s)$? [2pts]

solution: poles are $-1/5$ and $-1/10$. The zero is $1/2$. The plant is stable since the real parts of the poles are negative. The controller $G^{-1}(s)$ will be unstable (with pole at $1/2$) and therefore any uncertainty in the plant will give rise unstable tracking. (also the controller is not realizable (but not expected from students).

ii. Consider a proportional control design where $K(s)$ is equal to a constant k_c.

A. Find the sensitivity transfer function from r to e for this plant and show that higher values of k_c implies smaller tracking errors. [2pts]

solution: $S(s) = 1/(1 + G(s)K(s)) = 1/(1 + \frac{3k_c(1-2s)}{(5s+1)(10s+1)})$. As k_c is made bigger, the denominator of $|S(s)|$ is bigger and hence $|E(s)/R(s)| = |S(s)|$ is smaller.

B. Is the system stable for all positive values of k_c? Determine the range of values of k_c (given that $k_c > 0$) that guarantees stability. [3pts]

solution: No since, from root-locus rules as $k_c \to \infty$, one of the poles will end up at the zero which is at $1/2$, a positive real number.
The characteristic equation: $(5s+1)(10s+1) + 3k_c(1-2s) = 0$ which implies

$$50s^2 + (15 - 6k_c)s + (1 + 3k_c) = 0.$$

From Routh Hurwitz criteria, the stability conditions are $15 - 6k_c > 0$ and $1 + 3k_c > 0$. For $k_c > 0$ the second inequality is satisfied and the the first inequality is satisfied for $k_c < 2.5$. Therefore the range of values of k_c that guarantees stability $0 \le k_c < 2.5$.

iii. Find the gain $k_c^\star > 0$, the critical value at which the closed loop system becomes marginally stable (that is, when the closed loop poles are of the form $\pm j\omega^\star$). [2pts]

solution: The critical value from above analysis is $k_c^\star = 2.5$. The characteristic equation with $k_c^\star = 2.5$ becomes

$$50s^2 + 8.5 = 0,$$

which implies $s = \pm j\sqrt{8.5/50} = \pm j0.412$.

A. Find the corresponding frequency $\omega^\star$ and find the proportional-integral (PI) controller from the Ziegler and Nichols tuning rules given by

$$K(s) = \frac{k_c^\star}{2.2}\left(1 + \frac{0.6\omega^\star}{\pi s}\right).$$

[2pts]

solution: From above, $\omega^\star = 0.412$. Therefore

$$K(s) = \frac{2.5}{2.2}\left(1 + \frac{0.412 \times 0.6}{\pi s}\right) = 1.14\left(1 + \frac{1}{12.7s}\right).$$

B. Find the steady state error with this control design when the reference signal r is (i) a unit step and (ii) a unit ramp signal. [2pts]

solution: Now $E(s) = S(s)R(s)$ where $S(s) = 1/(1 + G(s)K(s))$. Therefore

$$E(s) = \left(\frac{12.7s(10s+1)(5s+1)}{12.7s(10s+1)(5s+1) + 1.14 \times 3(1-2s)(12.7s+1)}\right).$$

From final value theorem steady state error e_{ss} is given by $\lim_{s\to 0} sE(s) = sS(s)R(s)$. For unit step input, $R(s) = 1/s$, therefore $e_{ss} = S(0) = 0$. For unit ramp input, $R(s) = 1/s^2$, $e_{ss} = \left(\frac{12.7}{1.14\times 3}\right) = 3.725$.

Q2 [15pts]

Consider the systems with the following transfer functions:

$G_A(s) = \frac{1}{s^2+0.5s+1}$; $\omega_n = 1,\ \xi = 0.25;\ ss = 1$

$G_B(s) = \frac{4}{s^2+1s+4}$; $\omega_n = 2,\ \xi = 0.25;\ ss = 1$

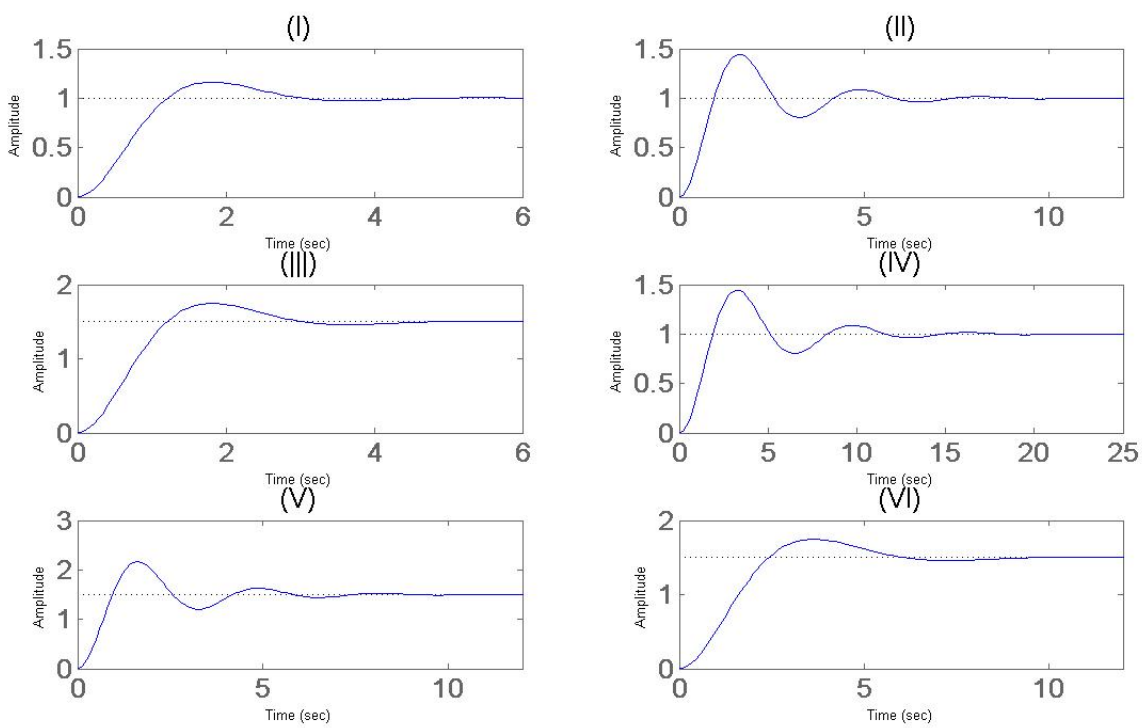
$G_C(s) = \frac{3}{2s^2+2s+2}$; $\omega_n = 1,\ \xi = 0.5;\ ss = 1.5$

$G_D(s) = \frac{6}{s^2+2s+4}$; $\omega_n = 2,\ \xi = 0.5;\ ss = 1.5$

$G_E(s) = \frac{4}{s^2+2s+4}$; $\omega_n = 2,\ \xi = 0.5;\ ss = 1$

$G_F(s) = \frac{12}{2s^2+2s+8;}$ $\omega_n = 2,\ \xi = 0.25;\ ss = 1.5$

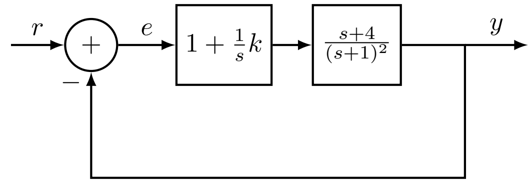
You are required to match these with the unit step responses shown below (*Hint: calculate the damping* ζ*, the natural frequency* ω_n *for each system and the corresponding steady state output values*).



Solution $A \to (IV),\ B \to (II),\ C \to (VI),\ D \to (III),\ E \to (I),\ F \to (V),$

Note that $A,\ B,\ E$ have steady state value 1.

Q3 [10pts]



Consider the feedback system shown in the figure, with transfer function $G(s) = \frac{s+4}{(s+1)^2}$ and controller of the form $C(s) = 1 + \frac{1}{s}k$.

(a) Find the range of k values that make the feedback system stable. [3pts]

Solution. When $k = 0$, the open-loop transfer function reduces to $G(s)$, which is stable. Now say that $k \neq 0$. The open-loop transfer function is given by

$$\left(1 + \frac{1}{s}k\right)\frac{s+4}{(s+1)^2} = \frac{s+k}{s}\frac{s+4}{s^2+2s+1} = \frac{s^2+(4+k)s+4k}{s^3+2s^2+s}.$$

When $k \neq 0$, the characteristic polynomial is given by

$$s^2 + (4+k)s + 4k + s^3 + 2s^2 + s = s^3 + 3s^2 + (5+k)s + 4k.$$

The Routh array is given by

$$\begin{array}{lcc} s^3: & 1 & 5+k \\ s^2: & 3 & 4k \\ s^1: & \frac{1}{3}\left(3(5+k) - 4k\right) & \\ s^0: & 4k & \end{array}$$

Thus, stability with $k \neq 0$ requires $k > 0$ and

$$3(5+k) > 4k \iff k < 15.$$

Thus, the feedback system is stable for all $k \in [0, 15)$.

(b) Can you find a k that achieves a steady-state error of 0.01 for a ramp input, $\frac{1}{s^2}$? [3pts]

Solution. No. The sensitivity function is given by

$$S(s) = \frac{1}{1 + C(s)G(s)} = \frac{1}{1 + \frac{s^2+(4+k)s+4k}{s^3+2s^2+s}} = \frac{s^2+2s^2+s}{s^3+3s^2+(5+k)s+4k}$$

The steady-state error for a ramp input is given by

$$\lim_{t\to\infty} s\frac{s^2+2s^2+s}{s^3+3s^2+(5+k)s+4k}\frac{1}{s^2} = \frac{1}{4k}$$

Thus, as k goes from 0 to 15, the steady state error goes from $+\infty$ to $1/60 \approx 0.017$. So, a steady state error of 0.01 is not achievable.

(c) Find a k that achieves an infinite gain margin and achieves a steady state error of below 0.1 for a ramp input, $1/s^2$. [4pts]

Solution. As long as the phase never drops below $-180°$, the system will have infinite gain margin. This can be achieved for any sufficiently small k. But in order to track a ramp input with error at most 0.1, the problem above shows that a gain k of at least 2.5 is needed.

For simplicity, try $k = 4$. Then the phase of the system is given by

$$\angle L(\jmath\omega) = \angle \frac{(j\omega+4)^2}{j\omega(j\omega+1)^2} = 2\tan^{-1}(\omega/4) - 2\tan^{-1}(\omega) - \frac{\pi}{2}.$$

For this system, the phase will decrease from -90° at $\omega = 0$, reach a local minimum, and then increase to back to -90° as $\omega \to \infty$. Let us find the local minimum:

$$\frac{d}{d\omega}\angle L(j\omega) = \frac{2}{1+(\omega/4)^2}\frac{1}{4} - \frac{2}{1+\omega^2} = 0$$

So we see that

$$1+\omega^2 = 4 + \omega^2/4 \implies \frac{3}{4}\omega^2 = 3 \implies \omega = 2$$

at the minimum.
From above, the phase at this frequency is

$$\angle L(j2) = 2\tan^{-1}(0.5) - 2\tan^{-1}(2) - \frac{\pi}{2} \approx -2.86 > -\pi.$$

Thus, the minimum phase is above -180° and the gain margin is infinite.